RRC
"Kurchatov Institute"
IAE-6342/2

S-wave $\pi\pi$ phase shifts and scattering lengths. The model-independent analysis.

V.N.Maĭorov, O.O.Patarakin

Russian Research Center "Kurchatov Institute",
Institute of General and Nuclear Physics,
pl. Kurchatova 1, 123182 Moscow, Russian Federation
E-mail: mvn@kiae.ru

Abstract

The model-independent analysis of the S- and P-wave $\pi\pi$ phase shifts was carried out. This analysis was based on the using of the Roy equations only and all available experimental data from the threshold up to dipion mass $m_{\pi\pi} = 1\ GeV$. As the results S-wave lengths were calculated: $a_0^0 = (0.212 \pm 0.015)\, m_\pi^{-1}$; $a_0^2 = (-0.043 \pm 0.010) m_\pi^{-1}$. The result obtained obviously confirm the standard ChPT version. Moreover, additional arguments were found in favor of the ratio of the S-wave phase shifts $\delta_0^0(s)$ and $\delta_0^2(s)$ being independent of energy from the threshold up to $m_{\pi\pi} = 900\ MeV$. The proportionality coefficient between the phase shifts η is equal to -4.66 ± 0.05.

PACS: 11.30.Qc; 11.55.Fv; 11.80.Et; 13.75.Lb; 14.40.Aq

Keywords: Roy equation; S-wave phase shift; S-wave scattering length; Chiral Perturbation Theory

1 Introduction

An investigation of the near-threshold parameters of the $\pi\pi$ interaction has acquired a special role due to emergence of QCD theories with a broken down chiral symmetry. During last years, Chiral Perturbation Theory (ChPT) [1, 2] and Generalised Chiral Perturbation Theory (GChPT) [3] were developed. Both these theories can describe the strong interactions at low energy. The determinative factor in these theories is the existence of vacuum condensates violating chiral symmetry. These theories having the same form of the effective Lagrangian differ from each other by value of quark condensate and light quark masses. The fact determining the choice of the version is that the S-wave $\pi\pi$ scattering lengths a_0^0 and a_0^2 are very sensitive to the parameters of the model and consequently are the key parameters for unambiguous determination of the scenario of chiral symmetry violation. In this way, ChPT predicts the value a_0^0=0.220 and GChPT $a_0^2 = 0.263$[1]. So, a reliable determination of the $\pi\pi$ lengths enables one to estimate the amount of chiral symmetry violation and to choose thereby an adequate version of the theory.

During some time, despite of large accumulated experimental material on scattering lengths, this choice has been difficult to be made. The matter is that the experiment K_{e4} [4] gave evidence in favor of GChPT, whereas most $\pi N \longrightarrow \pi\pi N$ experiments inclined rather to ChPT.

The aim of our program, begun in [5] and continued in [6, 7], was to choose a true chiral version without using additional constraints based on chiral theories. Therefore our calculation were based on the Roy equations only and all available experimental S- and P-wave phase shifts. In our work [6] very large uncertainties of the $\pi\pi$ lengths were obtained that prevented making unambiguous choice. In the next paper [7] the additional relation linking the S-wave phase shifts was used. This relation was received on the basis of the analysis of the S-wave behavior above the threshold only. Theses of chiral theories were not used at all. As the result, the accuracy of determination of S-wave lengths a_0^0 and a_0^2 was considerably improved by means of eliminating the correlation between them. The obtained lengths were in a good accordance with the standard ChPT version.

In the present work it will be shown that adding of the new data from the latest K_{e4} E865 [8] experiment to the base experimental data set used in [6, 7], makes it possible to improve considerably the accuracy of determination of a_0^0 and a_0^2 and for certain to choose, without using additional constraints, the scenario of chiral symmetry violation.

2 Roy equations

The using of the general principles of unitarity, analyticity and crossing symmetry is one of the seminal approaches to study $\pi\pi$ interaction. For $\pi\pi$ amplitudes, the integral equations known as "the Roy equations" [9-11] proved to be rather useful on this path. These equations determine the real parts of the partial wave amplitudes which satisfy the analyticity and crossing symmetry conditions in the $-4 < s < 60$ range in terms of $\pi\pi$ amplitude in the physical $4 < s < \infty$[2] region. The Roy equations combined with the unitarity relations constitute a system of non-linear singular integral equations. In deriving these equations, the dispersion relations with two subtractions at fixed four-momentum transfer **t** and the

[1]The S-wave scattering lengths a_0^0 and a_0^2 are given in m_π^{-1}

[2]Here and below, s is the Mandelstam variable, $s = m_{\pi\pi}^2/m_\pi^2$

crossing symmetry property of the scattering amplitudes were used. In present work the Roy equations were solved to get S-wave $\pi\pi$ lengths. All available experimental S- and P-wave phase shifts from the threshold up to $m_{\pi\pi} = 1\ GeV$ were used. And what's more the new high-accuracy data from latest K_{e4} E865 [8] experiment were added to the base experimental data set [12-23] used in [6].

For the S_0 wave description the phase shifts δ_0^0 obtained in the $\pi N \longrightarrow \pi\pi N$ and $\pi N \longrightarrow \pi\pi\Delta$ processes [12-17] were adopted. From [17] the values of the "down-flat" set was used only. In the region being studied, the "down-steep" solution coincides with "down-flat" one. Whereas the "up-flat" and "up-steep" versions cannot be described by a smooth curve and are strongly differed from the other data used. The results of the K_{e4} [4, 8] experiments were used also.

For the S_2 wave description the phase shifts δ_0^2 obtained in the $\pi^- p \longrightarrow \pi^-\pi^-\Delta^{++}$ [18-22] and $\pi^+ p \longrightarrow \pi^+\pi^+ n$ [23] processes were adopted. Precise values of the cross sections $\sigma_{\pi\pi}(s)$ near the threshold were obtained in [24]. This permitted to estimate the values δ_0^2 in this region under the assumption that phase shifts δ_0^0 are known.

As the result the phase shifts δ_0^2 and their uncertainties were calculated by using cross sections $\sigma_{\pi\pi}(s)$ and the values of δ_0^0 near the threshold from [4, 8, 14]. The resulting values are presented in the Table 1 (Appendix A). For the P-wave describing the results obtained in the $\pi^+\pi^- \longrightarrow \pi^+\pi^-$ and $\pi^\pm\pi^0 \longrightarrow \pi^\pm\pi^0$ channels [12,14-16] were used. For the case of the charged pions, the Roy equations are given by:

$$Re f_l^I(s) = \lambda_l^I(s) + \frac{1}{\pi}\int_4^{51} \Psi_l^I(x,s)\,dx + \varphi_l^I(s), \tag{1}$$

where $\Psi_l^I(x,s) = Im\, f_0^0(x) K_{1l}^I(x,s) + Im\, f_1^1(x) K_{2l}^I(x,s) + Im\, f_0^2(x) K_{3l}^I(x,s)$. Explicit expressions for the kernels $K_{jl}^I(x,s)$ are given in Appendix B. The corrections $\varphi_l^I(s)$ estimating the contributions from the higher waves ($l \geq 2$) and from the large mass region were adopted from [11].

$$\begin{aligned} \varphi_0^0(s) = 13\times10^{-5}(s^2-16) \pm \Delta\varphi_0^0; \quad \Delta\varphi_0^0 &= 5\times10^{-5}(s^2-16) \\ \varphi_0^2(s) = 13\times10^{-5}s(s-4) \pm \Delta\varphi_0^2; \quad \Delta\varphi_0^2 &= 6\times10^{-5}s(s-4) \end{aligned} \tag{2}$$

The subtraction terms $\lambda_0^I(s)$ are expressed in terms of the scattering lengths:

$$\lambda_0^0(s) = a_0^0 + \frac{s-4}{12}(2a_0^0 - 5a_0^2); \qquad \lambda_0^2(s) = a_0^2 - \frac{s-4}{24}(2a_0^0 - 5a_0^2) \tag{3}$$

We realized the same numerical method to solve the Roy equations as in [6] without using iterative procedures. Due to this approach the problem of convergence of the solutions was eliminated automatically and the process of calculation of scattering lengths a_0^0 and a_0^2 became absolutely clear. The solution of the Roy equations (1) comprised some steps. First, we performed fitting for each phase shift δ_l^I and obtained smooth curves adequately describing experimental data. In particular for the S-wave phase shifts expansion (4) was used:

$$\delta_0^I(s) = \frac{2}{\sqrt{s}}(C_1^I q + C_2^I q^3 + \cdots + C_m^I q^{2m-1}) \tag{4}$$

where $q = \frac{1}{2}\sqrt{s-4}$ - is c.m. pion momentum and C_k^I - are free parameters (I=0,2; $k = 1 \div 4$). We used m=4 because the increase of the number of terms of the series would not improve

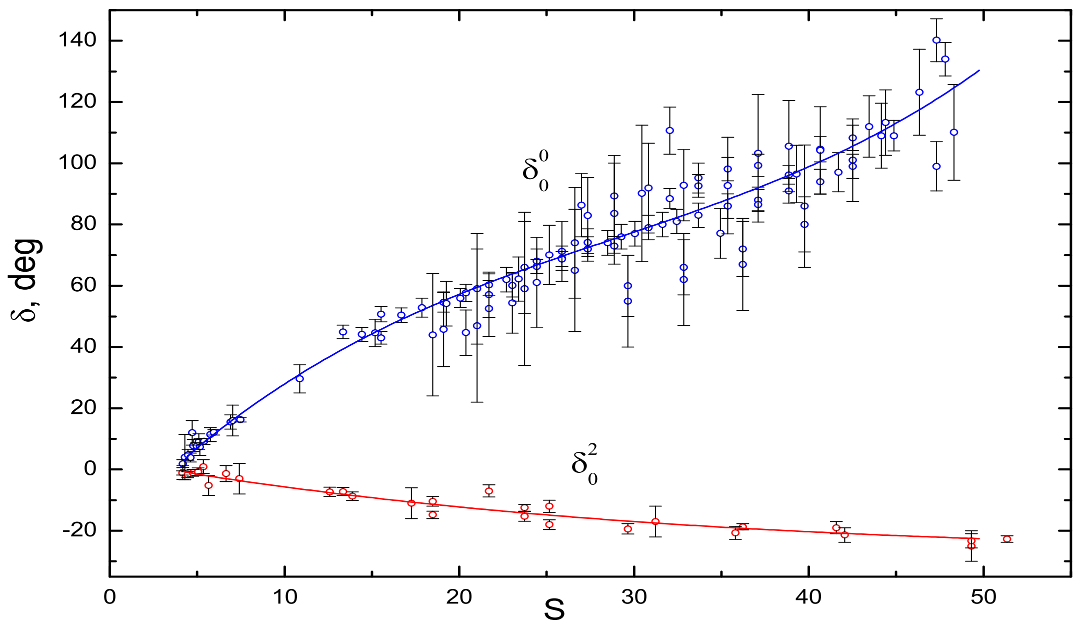

Figure 1: S-wave $\pi\pi$ phase shifts. The solid curves represent the result of fitting in terms of expression (4).

the accuracy of the smoothing. For the S_0 wave, when $n = 106$ experimental points were used, it was obtained: m=4, χ^2=137.76; m=5, χ^2=137.28. For the S_2 wave, when $n = 28$: m=4, χ^2=36.48; m=5, χ^2=36.36. Thus, the describing the S-wave phase shifts by means of the used polynomial is stable. Experimental values of phase shifts and fitting curves (4) are shown in Fig.1. In the present study, we assume, as in [6], that in the energy range considered the P-wave is determined by the rho-resonance almost completely.

On the second stage, the obtained smooth dependencies $\delta_l^I(s)$ were used as input for the Roy equations (1) and after integration, the subtraction terms $\lambda_l^I(s)$ were calculated.

The values of $Ref_0^I(s_i)$ were taken at each experimental point s_i where the phase shifts $\delta_0^I(s_i)$ were measured. By solving the Roy equations for each values of s_i, we obtained the values of the subtraction terms $\delta_0^I(s_i)$ and their statistical errors $\sigma_{\lambda_0^I}(s)$ from experimental data on the $\pi\pi$ phase shifts. This errors are determined ultimately by the errors of the phase shifts $\delta_0^I(s_i)$ and were calculated by means of the standard rule of propagation of errors. It should be emphasized that the expression for the uncertainties $\sigma_{\lambda_0^I}(s)$ does not contain the theoretical errors $\Delta\varphi_0^I(s)$, since they are not, generally speaking, statistical: they change the behavior of the function $\varphi_0^I(s)$ simultaneously for all s. Because of this, the theoretical corrections $\varphi_0^I(s)$ behave as random functions with respect to $\Delta\varphi_0^I(s)$. Therefore contribution of the uncertainties $\Delta\varphi_0^I(s)$ in the errors of the lengths a_0^0 and a_0^2 should be calculated separately.
At the conclusion stage we carried out fitting of the dependencies $\lambda_l^I(s_i)$ using terms (3) and determined the S-wave $\pi\pi$ lengths. Such approach enabled us to study in detail each isotopic channel of the Roy equations by evaluating the contribution of each phase shift δ_l^I in the resulting subtraction terms $\lambda_0^I(s)$ separately. It is in this way, it was found that

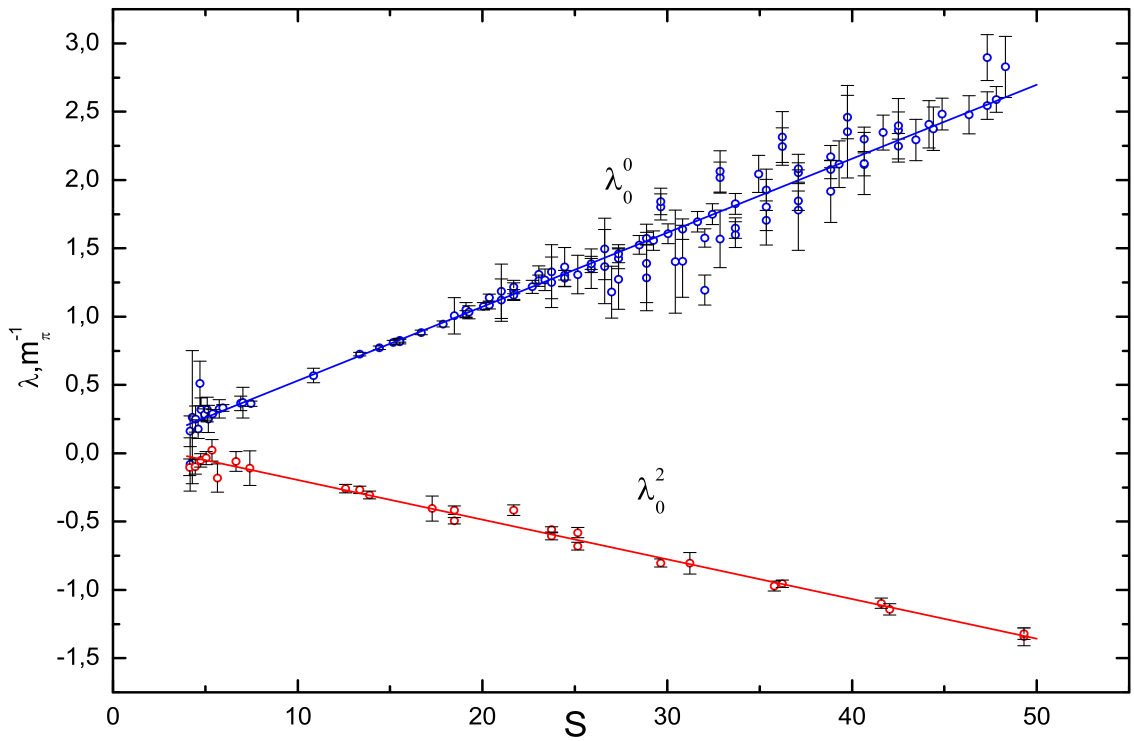

Figure 2: Subtraction terms $\lambda_0^0(s)$ and $\lambda_0^2(s)$. The straight lines represent the result of fitting in terms of expression (3).

the phase shifts δ_0^2, obtained in the " electronic experiment" [23], lead to the result which contradicts considerably the result obtained by processing the rest of the phase shifts δ_0^2 data base. Therefore, we did not use the phase shifts from [23] in the present study. This problem will be considered below. Now the solutions of the Roy equations obtained for each isotopic channel will be given. Hereinafter r is a factor of correlation between a_0^0 and a_0^2. In the isotopic channel I=0, it was obtained by fitting the subtraction term $\lambda_0^0(s_i)$:

$$a_0^0 = 0.207 \pm 0.009; \quad a_0^2 = -0.047 \pm 0.005; \quad r = 0.989 \tag{5}$$

χ^2/NDF=127/106. After taking into consideration the theoretical error $\Delta\varphi_0^0$ it was received:

$$a_0^0 = 0.207 \pm 0.015; \quad a_0^2 = -0.047 \pm 0.011; \quad r = 0.980 \tag{6}$$

In the isotopic channel I=2, it was got by fitting the subtraction term $\lambda_0^2(s_i)$:

$$a_0^0 = 0.295 \pm 0.042; \quad a_0^2 = -0.022 \pm 0.014; \quad r = 0.996 \tag{7}$$

χ^2/NDF=32.6/25. After taking into consideration the theoretical error $\Delta\varphi_0^2$, it was obtained:

$$a_0^0 = 0.295 \pm 0.094; \quad a_0^2 = -0.022 \pm 0.025; \quad r = 0.988 \tag{8}$$

The resulting subtraction terms $\lambda_0^I(s)$ with fitting straight lines (3) are shown in Fig.2.
The uncertainties of the $\pi\pi$ lengths a_0^0 and a_0^2 are defined both by the statistical errors of

values $\lambda_0^I(s_i)$, which are expressed through the uncertainties phase shifts by means of the standard rule of propagation of errors and by the theoretical uncertainties $\Delta\varphi_0^I(s)$.
It may seem that the lengths obtained from the isotopic channel I=2 are in accordance with the GChPT version, but it is not so. The large uncertainties of the obtained lengths do not permit to make any choice at all using this channel only. On the other hand, obviously, main information about the $\pi\pi$ lengths is contained in the term λ_0^0, because it concentrates in itself the overwhelming part of general statistics. We discuss the obtained solutions of the Roy equations for each isotopic channel in detail in order to show that these solutions ((6 and (8)) are in accordance with each other within the error limits and to demonstrate that the subtraction terms $\lambda_0^I(s)$ are really linear functions of s. For us, it is an additional proof that all the calculation steps in the solving the Roy equations and also all preliminary work comprising the fitting the phase shifts δ_l^I were carried out correctly.
The final result was obtained on the basis of the use of all available statistics, i.e., by both isotopic channels, I=0 and I=2:

$$a_0^0 = 0.212 \pm 0.015; \quad a_0^2 = -0.043 \pm 0.010 \tag{9}$$

with the correlation coefficient **r**=0.945. Now we can make a preliminary conclusion: the obtained results unambiguously witness in favor of the standard ChPT version and exclude the GChPT one. More detailed discussion of the obtained results will be provided in section 4.
In conclusion of the present section we shall demonstrate the results obtained by using the phase shifts from [23]. If the S_2-wave phase shifts are used only from [23] then the following is obtained:
a) δ_0^2 – Hoogland [23] only

$$I=0: \; a_0^0 = 0.209 \pm 0.015; \quad a_0^2 = -0.036 \pm 0.011; \; r = 0.978; \; \chi^2/NDF = 123/106 \tag{10}$$
$$I=2: \; a_0^0 = 0.140 \pm 0.071; \quad a_0^2 = -0.079 \pm 0.016; \; r = 0.986; \quad \chi^2/NDF = 32.5/5 \tag{11}$$

It is obvious that the results obtained in the different isotopic channels are contradictory for the parameter a_0^2. Moreover, a linearity test is not satisfied - the value of χ^2 in the channel I=2 (11) shows that the subtraction term $\lambda_0^2(s)$, obtained by using the phase shifts δ_0^2 from [23], is not a linear function of s.
When the both isotopic channels are used then the following is obtained:

$$a_0^0 = 0.163 \pm 0.015; \quad a_0^2 = -0.071 \pm 0.009; \; r = 0.904 \tag{12}$$

Thus, when the phase shifts δ_0^2 from [23] were used a concordance was absent both between the results obtained in the different isotopic channels as well as with the solution (9), obtained by using the rest of the phase shifts δ_0^2 data base, taken from [18-22].
The using of the united phase shifts δ_0^2 data base from [18-23] does not change the situation in principle by force of statistical domination of the phase shifts from [23].
b) δ_0^2 – Hoogland [23] + all the rest

$$I=0: \; a_0^0 = 0.209 \pm 0.015; \quad a_0^2 = -0.039 \pm 0.011; \; r = 0.979; \; \chi^2/NDF = 125/106 \tag{13}$$
$$I=2: \; a_0^0 = 0.182 \pm 0.075; \quad a_0^2 = -0.064 \pm 0.018; \; r = 0.992; \quad \chi^2/NDF = 92/32 \tag{14}$$

It was got by using both channels:

$$a_0^0 = 0.177 \pm 0.015; \quad a_0^2 = -0.064 \pm 0.010; \; r = 0.921 \tag{15}$$

It should be noted that adding the phase shifts δ_0^2 from [23] in the channel I=0, which is the main source of the information about a_0^0 and a_0^2, leads to the systematic increase of the value a_0^2 (13).

3 Correlation between $\delta_0^0(s)$ and $\delta_0^2(s)$

In the previous section it was shown that the strongly correlated S-wave $\pi\pi$ lengths in the result of solution of the Roy equations were obtained. Such correlation, with $r \cong 1$, implies that the values a_0^0 and a_0^2 are related by a linear dependence. But this fact signifies that the phase shifts $\delta_0^0(s)$ and $\delta_0^2(s)$ by force of the near threshold expansion $\delta_0^I(s) \propto a_0^I\, q$ must be related by a linear dependence too in some energy region near the threshold. We do not know only the range of this region . We analyzed the ratio $\xi(s) = \delta_0^0(s)/\delta_0^2(s)$ for the available experimental data to study this problem. No evident dependence on s was found in the behavior of $\xi(s)$ from the threshold up to s=42, i.e., up to $m_{\pi\pi}$=900MeV (Fig.3).
In such a way, the simplest 0-hypothesis to verify is the hypothesis about proportionality phase shifts in some area above the threshold. As the phase shifts $\delta_0^0(s_i)$ and $\delta_0^2(s_j)$ were measured mainly at different energy values, the smoothed curve (Fig.1) representing the fitting function (4) was used for calculation of the phase shifts δ_0^2 at the points $s = s_i$, where the phase shifts δ_0^0 were measured. Thus, the ratio of the S-wave phase shifts was calculated as $\xi(s_i) = \delta_0^0(s_i)/\delta_0^2(s = s_i)$.
The uncertainties σ_ξ were calculated by the standard rule of propagation of errors and finally they were defined both by errors of phase shifts $\delta_0^0(s_i)$ and $\delta_0^2(s_j)$. It was calculated by fitting $\xi(s) \equiv \eta$–const, for interval $s = 10 \div 42$: $\eta = -4.66 \pm 0.05$; χ^2/NDF=78/82. The fitting $\xi(s) \equiv \eta$–const for the interval $s = 4 \div 42$ gives naturally the same value of η, because the statistical weights of the points near the threshold are insignificant. In general, large uncertainties of the values $\xi(s)$ near the threshold (Fig.3) are caused by the fact that the phase shifts $\delta_0^0(s)$ and $\delta_0^2(s)$ have large relative errors in that region.
So, the proposed 0-hypothesis is confirmed by means of the statistical proof. And consequently we can conclude that within the present accuracy of the experimental data the ratio of S-wave phase shifts does not depend on the energy in the wide enough region. Thus, for this energy region, i.e., for $s = 10 \div 42$, the relation take place:

$$\delta_0^0(s) = \eta\, \delta_0^2(s) \tag{16}$$

where $\eta = -4.65 \pm 0.05$. So, as stated above, from the fact of strong correlation of the $\pi\pi$ lengths follows linear dependence of the phase shifts near the threshold . Then we found the proportionality between $\delta_0^0(s)$ and $\delta_0^2(s)$ in some region above the threshold: $s = 10 \div 42$. Our only proposal based on these facts is that we deal with the same proportionality. I.e., we believe that the relation (16) is true from the threshold up to s=42. Hence, in force by the near threshold expansion $\delta_0^I(s) \propto a_0^I\, q$, the new constraint on scattering lengths follows:

$$a_0^0 = \eta\, a_0^2 \tag{17}$$

So, an opportunity appears using the constraint (17) to eliminate the correlation between a_0^0 and a_0^2 in the process of the subtraction terms fitting. As the result, the solution was obtained, which we denote as "η-solution":

$$a_0^0 = 0.211 \pm 0.005; \quad a_0^2 = -0.0454 \pm 0.0010 \tag{18}$$

Here it should be emphasized that η-solution is in accord with the solution (9) obtained without using additional constraints. It is a very important point. This signifies that the additional constraint (17) relating the $\pi\pi$ lengths does not correct the Roy equations but eliminates the correlation only, when subtraction terms are fitting.

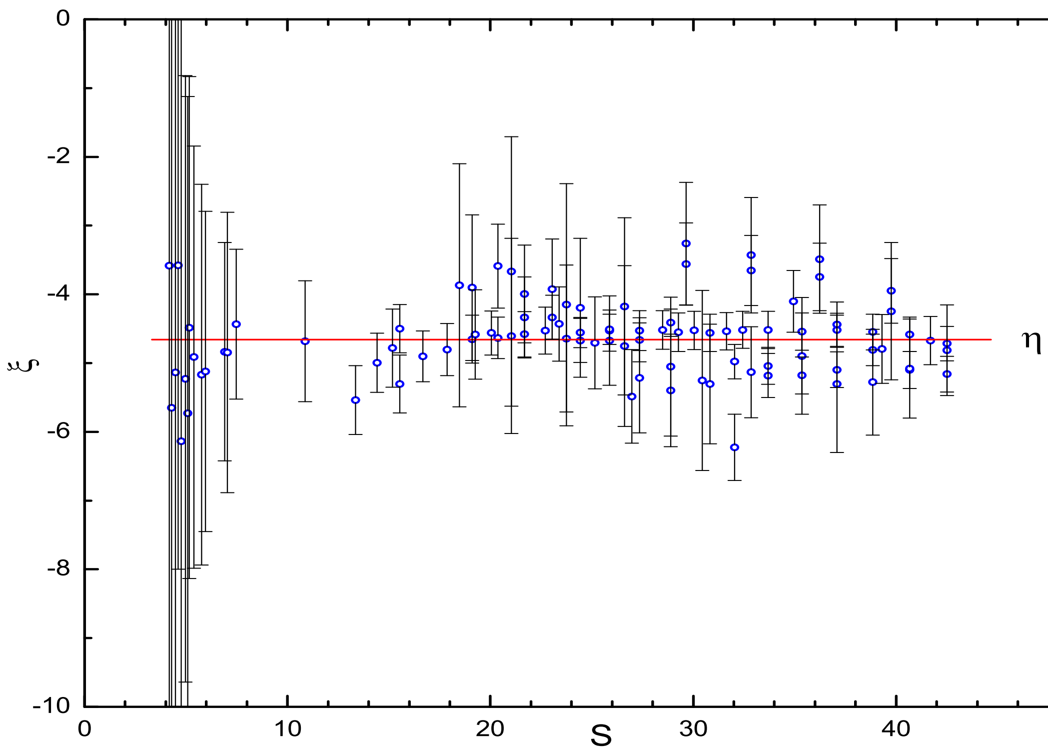

Figure 3: The ratio of the S-wave phase shifts $\xi(s) = \delta_0^0(s)/\delta_0^2(s)$. The straight line represents the constant $\eta = -4.66$.

In this sense the condition (17) is a new independent constraint on S-wave $\pi\pi$ lengths. We stress than the process of obtaining the η-solution and the solution (9) is the same right up to calculating subtraction terms $\lambda_0^I(s)$ inclusive. The difference between them consists in using the constrain (17) for obtaining the η-solution on the fitting step. The solution (9) was obtained without using of any additional constraints. The obtained results are presented in Fig.6 and Fig.7.

4 Discussion and Summary

Let us analyse obtained results in more detail. We should start by comparison of our result (9) with the theoretical prediction received in [27], in which ChPT calculations were supplemented with the phenomenological representations based on the Roy equations [25]:

$$a_0^0 = 0.220 \pm 0.005; \quad a_0^2 = -0.0444 \pm 0.0010 \tag{19}$$

These results are in good accord with each other for both parameters a_0^I within error limits. Hence our result (9) certainly witnesses in favor of the standard ChPT version and excludes GChPT one, with $a_0^0 = 0.263$. Thus, the problem of choosing the true ChPT version, in our opinion, is solved.

But it is possible to put a more tough question: whether there is statistically significant conformity between the theoretical result (19) and the result of the model–independent

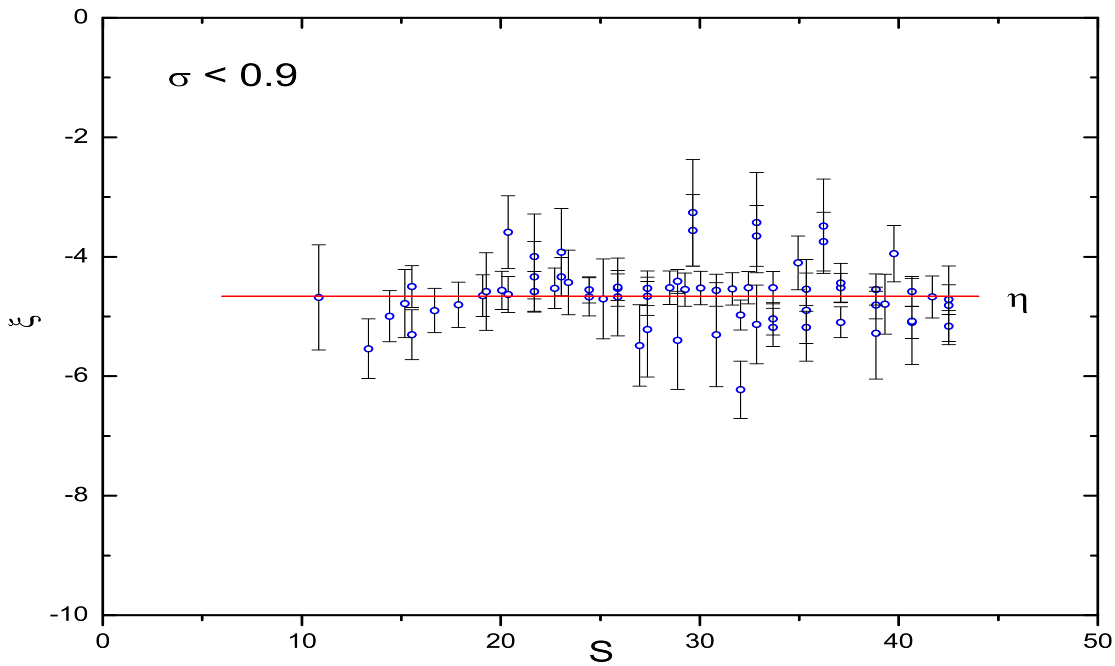

Figure 4: The ratio of the S-wave phase shifts $\xi(s) = \delta_0^0(s)/\delta_0^2(s)$ after the filtration, $\sigma_\xi < 0.9$. The straight line represents the constant $\eta = -4.66$.

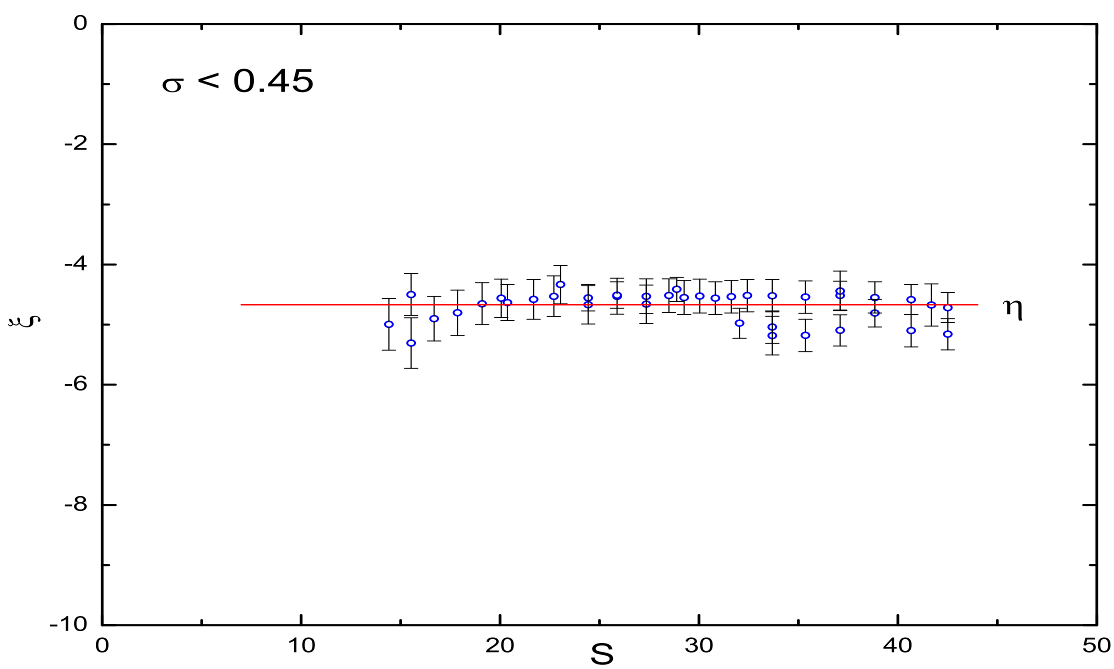

Figure 5: The ratio of the S-wave phase shifts $\xi(s) = \delta_0^0(s)/\delta_0^2(s)$ after the filtration, $\sigma_\xi < 0.45$. The straight line represents the constant $\eta = -4.66$.

analysis (9)? It is seen that 1σ contour ellipses do not intersect (Fig.6). Is the hypothesis true that these results are statistically consistent one with the other or may be there is a significant statistical discrepancy of these results?
Here it is necessary to take into account one feature which distinguish comparison of the results in an one-dimensional case and in a plane. In a plane the probability of a random variable to get inside a 1σ contour ellipse is equal P=0.39. Certainly it is not enough to draw final conclusions. Therefore it is more correct to compare 2σ contour ellipses. The probability to get in such an ellipse is equal P=0.865. The ChPT solution (19) gets in the border of the 2σ contour ellipse (9) (Fig.6). I.e., we have to reject the hypothesis that the results (9) and (19) are consistent with probability 13.5%. This probability is very large.
All this taken together forces us to come to a conclusion, that we do not have sufficient base to reject a hypothesis about the statistical agreement of the results (9) and (19). Thus, we come to the conclusion, that the solutions (9) and (19) are statistically consistent and do not contradict each other.
Let us carry out comparison with other works in which the results of experiment K_{e4} E865 for calculation of S-wave $\pi\pi$ lengths were used. In the work [8], where the final results of this experiment were presented, it was received without using of the additional relations linking a_0^0 and a_0^2:

$$a_0^0 = 0.203 \pm 0.033 \pm 0.004_{syst}; \quad a_0^2 = -0.055 \pm 0.023 \pm 0.003_{syst} \tag{20}$$

I.e., we have full conformity with our result (9) within the limits of errors (Fig. 7). In the work [25] the position and the borders of the area in the plane (a_0^0, a_0^2) in which S-wave lengths are consistent with the Roy equations solution and the available experimental data on $\pi\pi$ phase shifts above 0.8 GeV were specified. It was received for the central curve of this area:

$$a_0^2 = -0.0849 + 0.232 a_0^0 - 0.0865 (a_0^0)^2 \ [\pm 0.0088] \tag{21}$$

The value given in brackets defines the width of the band. In Fig.6 and Fig.7 this band is designated as UB (universal band).
In the work [26] the calculations done in [25] were repeated with some changes and practically the same parameters describing UB were received. Further, using the obtained parametrization and the experimental data including the data [8], the authors received:

$$a_0^0 = 0.228 \pm 0.013; \quad a_0^2 = -0.0380 \pm 0.0044 \tag{22}$$

with the factor of correlation **r**=0.799. In the work [8] using UB [25] as the additional constraint close results were received.
The results obtained in [8] and [26] are given in Fig.7. The solution (22) gets in our 2σ contour ellipse as well as our solution (9) gets in 2σ contour ellipse of the solution (22). Thus, it is possible to state that the results (9) and (22) do not contradict one another.
Let us consider the problem of stability of the received solution (9) concerning the procedure of experimental data selection, i.e., $\pi\pi$ phase shifts, which in our method of the solution of the Roy equations are utilized as input. Stability of the solutions versus variations of the initial data is an important indicator of reliability of the method of the solution and consistency of the initial data. We have shown above that the use of the data from [23] leads to contradictious results. Further, the results of an expanded analysis are presented.

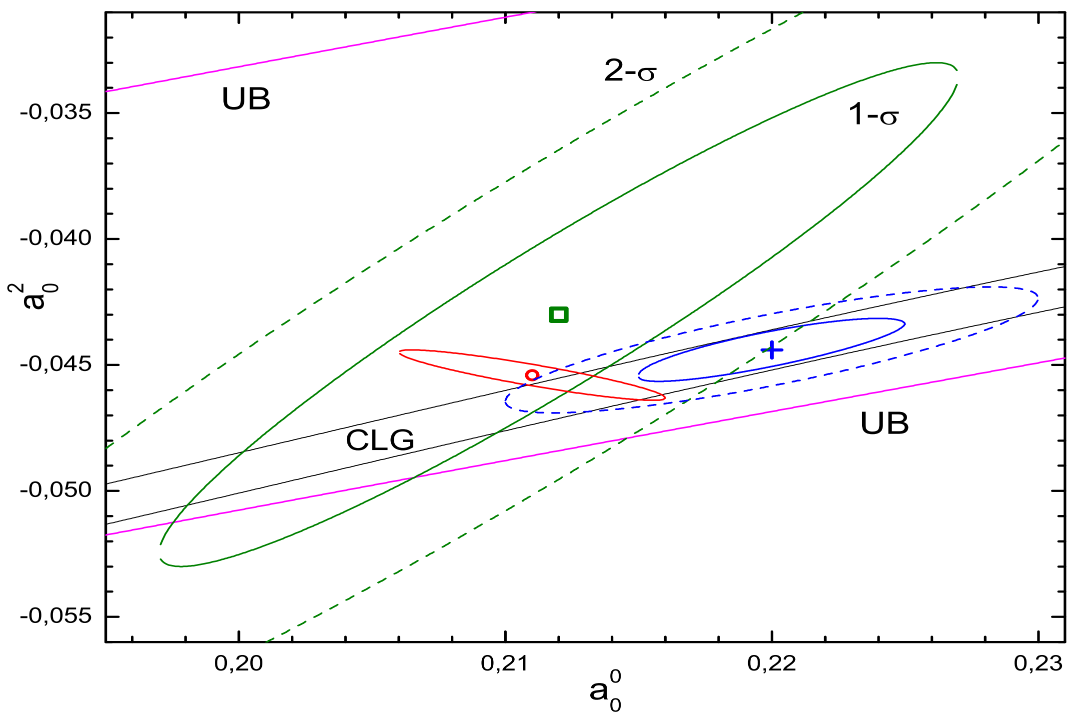

Figure 6: (Color online) The S-wave $\pi\pi$ lengths. The olive ellipse with the centre as a square indicates the solution (9); the solid line - 1σ ellipse, the dashed one - 2σ. The red ellipse with the centre as a circle represents η-solution (18). The blue ellipse with the centre as a cross shows the ChPT result [27]; the solid line - 1σ ellipse, the dashed one - 2σ. The straight lines marked UB indicates the area allowed for S-wave lengths [25]. The strip marked CLG is the range corresponding the chiral constraint [28].

1) Change of the data sets used.

a) The solution of the Roy equations without the phase shifts δ_0^2 which calculated on the basis of the cross sections received in [24] (Tabl. 1):

$$a_0^0 = 0.213 \pm 0.015; \quad a_0^2 = -0.044 \pm 0.011; \quad r = 0.961 \tag{23}$$

b) Calculation of S-wave lengths without the data from the work [14]:

$$a_0^0 = 0.208 \pm 0.015; \quad a_0^2 = -0.045 \pm 0.010; \quad r = 0.944 \tag{24}$$

2) Change of the degree of the fitting S-wave phase shifts polynomial (4):
For m=5 in the formula (4) we receive:

$$a_0^0 = 0.211 \pm 0.013; \quad a_0^2 = -0.044 \pm 0.009; \quad r = 0.918 \tag{25}$$

Comparison of the results (23-25) with the above solution (9) shows, that the criterion of the stability for the given solution is satisfied.
Let us proceed to the discussion of observable proportionality of S-wave phase shifts. It may

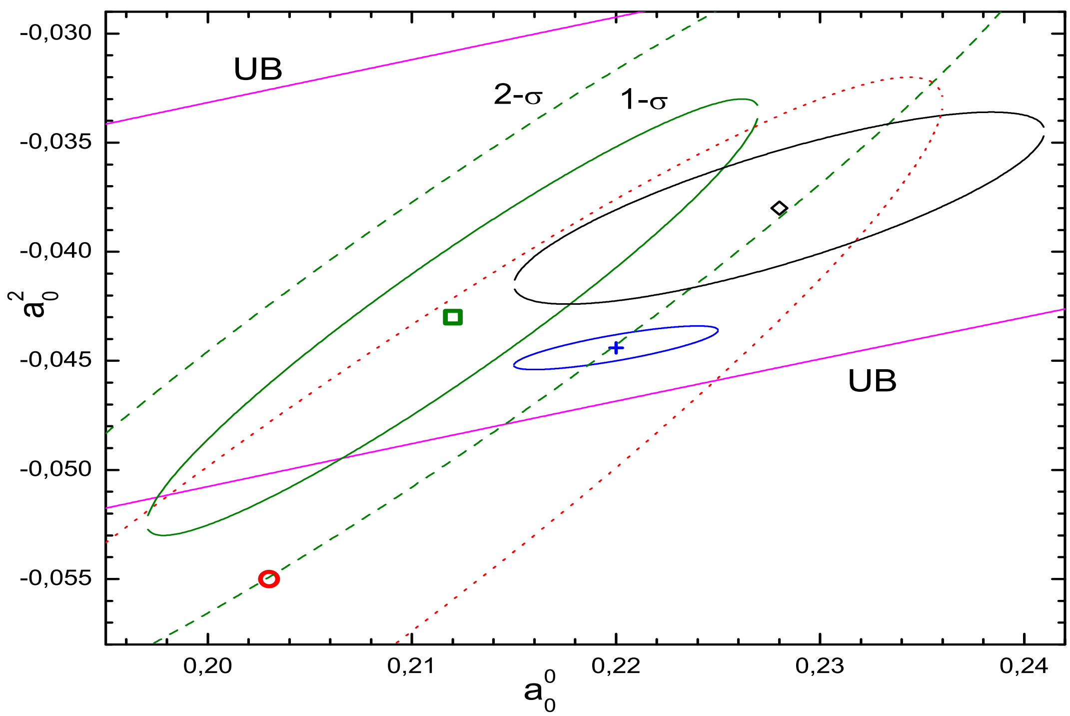

Figure 7: (Color online) The S-wave $\pi\pi$ lengths. The olive ellipse with the centre as a square indicates the solution (9); the solid line - 1σ ellipse, the dashed one - 2σ. The black ellipse with the centre as a rhomb represents the result [26]. The red dotted ellipse with the centre as a circle represents the result [8]. The blue ellipse with the centre as a cross shows the ChPT result [27].

seem, that the values $\xi(s_i)$ in Fig.3 have a wide scatter and, therefore, can be described not only by a constant, but also by some class of smooth functions of s. But these doubts are based on visual illusion. The point is that the values $\xi(s_i)$ with the large errors σ_ξ form "a cloud" which masks true dependence. These points have small statistical weights and do not give contribution to η value. We have carried out a filtration leaving only the points with errors less than given, i.e., with $\sigma_\xi < \sigma_k$ where σ_k lay in the range $1 \div 0.3$. The remaining after the filtration values of $\xi(s_i)$ were fitted by a constant. The result: all received η_k lay in the range $(-4.66 \div -4.68)$ and have errors $\sigma_\eta = 0.05$ and goodness-of-fit test is satisfied: $\chi^2 < N_k$, where N_k - number of points $\xi(s_i)$ after the filtration with parameter σ_k. I.e., all sets of points ξ_i after the filtration are well described by a constant. The results are given in Fig. 3-5. Fig. 3 - the values $\xi(s_i)$ without the filtration, Fig.4 - $\sigma_k = 0.9$, Fig.5 - $\sigma_k = 0.45$. One can easily see that after rejection of the points with large errors, the remaining points more and more concentrate near the straight line. Thus, if the 0-hypothesis is that values $\xi(s_i)$ are the constant within the considered region, this hypothesis are proved both statistically and visually.

Additionally, a linear function was utilized for fitting $\xi(s_i)$ also to find any dependence of ξ on s, if it exists nevertheless. The linear function may be represented as $f(s = \eta' + b\,(s-4)$,

where η' and b are free parameters. It was obtained as the result of fitting for $s = 10 \div 42$:

$$\eta' = -4.51 \pm 0.15; \quad b = -0.006 \pm 0.006 \tag{26}$$

χ^2/NDF=77/81. As $|b| \leq \sigma_b$, there is no reason to believe, that the hypothesis about the linear dependence $\xi(s)$ is confirmed.

Let us consider the problem of influence of the η errors on the $\pi\pi$ lengths errors obtained by the solving of the Roy equations (η-solution (18)). It may seem that the sizes of a_0^I errors are small because the sizes σ_η are small. But it is not so. The basic contribution to errors of the values a_0^0 and a_0^2, in this case, is brought by theoretical uncertainties in the Roy equations $\Delta\varphi_0^I(s)$. Role of the additional constraint (17) is only to eliminate the correlation between a_0^0 and a_0^2 in the process of fitting of subtraction terms $\lambda_0^I(s)$. To show it we increase σ_η four times, i.e., we used $\sigma_\eta = 0.2$. In result the following η-solution is obtained:

$$a_0^0 = 0.211 \pm 0.0052; \quad a_0^2 = -0.0454 \pm 0.0016 \tag{27}$$

The solution (27) shows a weak dependence on σ_η.

In the works [28, 29] it was shown that the width of the allowed area in the (a_0^0, a_0^2) plane can be reduced considerably by using the additional chiral constraint imposed on S-wave lengths. This constraint links the combination $(2a_0^0 - 5a_0^2)$ with the scalar pion radius $< r_s^2 >$. In the result of utilizing of this constraint it was received:

$$\Delta a_0^2 = 0.236\Delta a_0^0 - 0.61(\Delta a_0^0)^2 - 9.9(\Delta a_0^0)^3 \; [\pm 0.0008] \tag{28}$$

where $\Delta a_0^0 = a_0^0 - 0.22$; $\Delta a_0^2 = a_0^2 + 0.0444$. In Fig.6 this narrow strip is denoted as CLG. From this figure it follows that η-solution (18) lay practically in the border of the CLG-band and half of 1σ contour ellipse overlaps this band. Also η-solution lays practically in the border of 2σ contour ellipse of the ChPT-solution (19). Owing to all aforementioned, one may conclude that η-solution (18) received under the additional condition (17) is consistent both with the chiral CLG constraint (28) within the 1σ level and with the ChPT-solution (19) within the 2σ level .

It is natural to compare the value of $\eta = -4.66 \pm 0.05$ received in the present work with $\eta_{ChPT} = a_0^0/a_0^2$, which follows from the chiral theory. Only it is necessary to take into account that the calculation of the ratio of S-wave $\pi\pi$ lengths should be carried out in view of their correlation. From values a_0^0, a_0^2 and $2a_0^0 - 5a_0^2$ obtained within ChPT framework [27, 29], one may estimate the factor of correlation between a_0^0 and a_0^2. It is equal 0.8. In view of it for the S-wave $\pi\pi$ lengths ratio it was received $\eta_{ChPT} = -4.95 \pm 0.21$. The difference from the η value received by us is slightly more than one σ.

Summarizing the main results of the present study, one may say that the solutions received by us and other authors [8, 26] (Fig.7) are grouped near the ChPT-solution [27] and are consistent both with each other and with this ChPT solution. Thus, we believe that the problem of choosing of the scenario of chiral symmetry violation is solved. The available mismatch on the 1σ level both among the considered Roy equations solutions and among these solutions and the theoretical prediction [27] may be caused by the fact that we used non identical sets of the experimental data and different methods of the Roy equations solution. Therefore, it seems that prior to search for the physical reasons of such divergence, it is necessary to come to an agreement about using of uniform experimental data base. Also it is desirable to organize the procedure of the Roy equations solution in such a way that enables to check both individual solutions in every isotopic channel and monitor influence

of various errors (statistical, systematic, theoretical, errors from additional constraints) on the resulting errors of the S-wave $\pi\pi$ lengths. May be that such unification of the initial data and more detailed control of the course of the solution will allow to reduce the existing discrepancy.

This work was supported in part by the Russian Foundation for Basic Research (project no. 00-02-17852).

Appendix A

Table 1.

s	E, MeV	δ_0^2, deg	$\sigma(\delta_0^2)$, deg
4.15	284.3	-1.11	0.67
4.45	294.3	-1.75	0.96
4.75	304.2	-1.05	1.11
5.05	313.7	-0.71	1.26
5.35	322.8	0.93	2.30
5.65	331.8	-5.20	3.27

Appendix B

$$K^0_{10} = \frac{s-4}{(x-s)(x-4)} + \frac{2}{3x}\left[\frac{x}{s-4} ln\left(\frac{x+s-4}{x}\right) - 1\right] - \frac{2(s-4)}{3x(x-4)}$$

$$K^0_{20} = \frac{3}{x}\left\{2\left(1+\frac{2s}{x-4}\right)\left[\frac{x}{s-4} ln\left(\frac{x+s-4}{x}\right) - 1\right] + \frac{s-4}{x-4}\right\}$$

$$K^0_{30} = \frac{5}{3x}\left\{2\left[\frac{x}{s-4} ln\left(\frac{x+s-4}{x}\right) - 1\right] + \frac{s-4}{x-4}\right\}$$

$$K^1_{11} = \frac{1}{3}\left\{\frac{4}{s-4}\left[\left(\frac{1}{2}+\frac{x}{s-4}\right) ln\left(\frac{x+s-4}{x}\right) - 1\right] - \frac{s-4}{3x(x-4)}\right\}$$

$$K^1_{21} = \frac{s-4}{(x-s)(x-4)} + \frac{6}{s-4}\left(1+\frac{2s}{x-4}\right)\left[\left(\frac{1}{2}+\frac{x}{s-4}\right) ln\left(\frac{x+s-4}{x}\right) - 1\right] - \frac{3(s-4)}{2x(x-4)}$$

$$K^1_{31} = -\frac{5}{3}\left\{\frac{2}{s-4}\left[\left(\frac{1}{2}+\frac{x}{s-4}\right) ln\left(\frac{x+s-4}{x}\right) - 1\right] - \frac{s-4}{6x(x-4)}\right\}$$

$$K^2_{10} = \frac{1}{3x}\left\{2\left[\frac{x}{s-4} ln\left(\frac{x+s-4}{x}\right) - 1\right] - \frac{s-4}{x-4}\right\}$$

$$K^2_{20} = -\frac{3}{x}\left\{\left(1+\frac{2s}{x-4}\right)\left[\frac{x}{s-4} ln\left(\frac{x+s-4}{x}\right) - 1\right] - \frac{s-4}{2(x-4)}\right\}$$

$$K^2_{30} = \frac{s-4}{(x-s)(x-4)} + \frac{1}{3x}\left[\frac{x}{s-4} ln\left(\frac{x+s-4}{x}\right) - 1\right] - \frac{5(s-4)}{6x(x-4)}$$

References

[1] J.Gasser and H.Leutwyler, Phys. Lett. B125 (1983)321; 325

[2] J.Bijnens et al., Phys. Lett. B374(1996)210

[3] M.Knecht et al., Nucl. Phys. B457 (1995) 513; B471(1996)445

[4] L.Rosselet et al., Phys. Rev. D15 (1977)574

[5] O.O.Patarakin, V.N.Tikhonov, K.N.Mukhin, Nucl. Phys. A598(1996)335

[6] V.N.Maĭorov, O.O.Patarakin, V.N.Tikhonov, Yad. Fiz. 63(2000)1699 [Phys. At. Nucl. 63 (2000) 1612]

[7] V.N.Maĭorov, O.O.Patarakin, Preprint IAE-6274/4 (Moscow, 2003); hep-ph/0308162

[8] S.Pislak et al., Phys. Rev. D67 (2003) 072004

[9] S.M.Roy, Phys. Lett. B36 (1971) 353

[10] J.L.Basdevant, C.D.Frogatt and L.Petersen, Nucl. Phys. B72(1974)413

[11] M.R.Pennington, S.D.Protopopescu, Phys. Rev. D7 (1973)1429

[12] S.D.Protopopescu et al., Phys. Rev. D7 (1973) 1279

[13] A.Engler et al., Phys. Rev. D10 (1974) 2070

[14] E.A.Alekseeva et al., Zh. Eksp. Teor. Fiz. 82(1082)1007 [Sov. Phys. JETP 55(1982)591]

[15] E.A.Estabrooks and A.D.Martin, Nucl. Phys. B79 (1974)301

[16] J.Carroll et al., Phys. Rev. D10(1974)1430

[17] R.Kaminski et al., Z. Phys. C74(1997)79

[18] J.Losty et al., Nucl. Phys. B69 (1974)185

[19] E.Colton et al., Phys. Rev. D3 (1971)2028

[20] D.Cohen et al., Phys. Rev. D7 (1973) 661

[21] N.B.Durusoy et al., Phys. Lett. B45 (1973)517

[22] G.V.Beketov et al. Yad. Fiz. 19 (1974) 1032 [Sov. J. Nucl. Phys. 19 (1974) 528]

[23] W.Hoogland et al., Nucl. Phys. B126 (1977) 109

[24] M.Kermani, O.Patarakin, G.R.Smith et al., Phys. Rev. C58 (1998)3431

[25] B.Ananthanarayan et al., Phys. Rep. 353 (2001) 207; hep-ph/0005297

[26] S.Descotes et al., Eur. Phys. J. C24 (2002) 469; hep-ph/0112088

[27] G.Colangelo, J.Gasser and H.Leutwyler, Phys. Lett. B488(2000)261; hep-ph/0007112

[28] G.Colangelo, J.Gasser and H.Leutwyler, Phys. Rev. Lett. 86(2001)5008; hep-ph/0103063

[29] G.Colangelo, J.Gasser and H.Leutwyler, Nucl. Phys. B603 (2001)125; hep-ph/0103088